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Exact Analytic Solution of the Simplified Telegraph Model of Propagation and Dissipation of Excitation Fronts

H. A. Abdusalam[1,2]

The telegraph equation is more suitable than ordinary diffusion equation in modeling reaction diffusion in several branches of sciences (E. Ahmed, H. A. Abdusalam, and E. S. Fahmy, 2001, *Int. J. Mod. Phys. C* **12**(5), 717; E. Ahmed and H. A. Abdusalam, 2004, *Chaos, Solitons and Fractals* **22**, 583; H. A. Abdusalam and E. S. Fahmy, 2003, *Chaos, Solitons and Fractals* **18**, 259; Abdusalam, 2004 *Appl. Math. Comp.* **157**, 515). An excitation wave in cardiac tissue fails to propagate if the transmembrane voltage at its front rises too slow and does not excite the tissue ahead of it. Then the sharp voltage profile of the front will dissipate, and subsequent spread of voltage will be purely diffusive. This mechanism is impossible in FitzHugh–Nagumo type system (V. N. Biktashev, 2003, *Int. J. Bifarcation and Chaos* **13**(12), 3605). Biktashev suggested a simplified mathematical model for this mechanism and in the present work we generalize this model to telegraph system. Our generalized telegraph model has exact traveling front solutions and we show the effect of the time delay on the velocity and we show that, the post-front voltage depends on two parameters in which one of them is the time delay.

KEY WORDS: diffusion; telegraph reaction diffusion system; Nagumo equation; FitzHugh–Nagumo; excitation fronts.

1. INTRODUCTION

Hodgkin and Huxley (1952) have proposed a mathematical model of the electric action of the giant squid axon. After this model a large family of models describing other related phenomena were studied, e.g., excitability of heart muscle, this mathematical models are rather complicated and mostly studied numerically.

FitzHugh (1961) and Nagumo *et al.* (1962) suggested a simplified analogue of the Hodgkin and Huxley equations:

[1] Mathematics Department, Faculty of Science, Cairo University, Giza, Egypt.
[2] King Saud University, College of Science, Mathematics Department, Kingdom of Saudi Arabia, P.O. Box 2455, Riyadh 11451; e-mail: hosny@operamail.com.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \epsilon_u\left(u - \frac{u^3}{3} - v\right)$$
$$\frac{\partial v}{\partial t} = \epsilon_v(u + \beta - \gamma v) \tag{1}$$

where u corresponds to the transmembrane voltage (Biktashev, 2003) and v represents all other slow variables. Nagumo *et al.* (1962) have demonstrated that system (1) describes propagating pulses similar to those in Hodgkin and Huxley system, and allows a great deal of analytical and qualitative study (Abdusalam, 2004).

FitzHugh–Nagumo (FHN) system and its modifications served well as simple but reasonable models of excitation propagation in nerve, heart muscle, and other biological excitable media (FitzHugh, 1969).

If $\epsilon_u \ll \epsilon_v$, system (1) is simplified to the following FHN systems (Murray, 2003; Rinzel and Terman, 1982; Keener, 1980)

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(u, v)$$
$$\frac{\partial v}{\partial t} = \epsilon g(u, v), \quad 0 < \epsilon \ll 1 \tag{2}$$

The simplified model of excitation front for the realistic models descendants of Hodgkin–Huxley model, constructed by Biktashev is given by the following system of equations (Biktashev, 2003),

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + H(u - 1)v \tag{3}$$
$$\frac{\partial v}{\partial t} = \frac{1}{a}(H(-u) - v) \tag{4}$$

where $H(u)$ is the Heaviside step function and a is a one-dimensionless parameter.

A traveling wavefront solution of system (3) and (4) has to have

$$u(x, t) = U(z), \quad v(x, t) = V(z); \quad z = x + ct \tag{5}$$

where c is the wave speed: with $c > 0$ these represent waves traveling to the left. Substituting from (5) into (3) and (4) gives

$$cU' = U'' + H(U - 1)V \tag{6}$$
$$cV' = \frac{1}{a}(H(-U) - V) \tag{7}$$

with the auxiliary conditions:

$$U(-\infty) = -\alpha < 0; \qquad U(+\infty) = \varpi > 1$$
$$V(-\infty) = 1; \qquad V(+\infty) = 0. \tag{8}$$

and the internal boundary conditions at $z = 0$ and $z = z_1 > 0$ are given by

$$U(0) = 0; \quad U(z_1) = 1. \tag{9}$$

where $U(z) \in C^1$ and $V(z) \in C^0$.

By using the internal and auxiliary conditions (8) and (9), the simplified model (3) and (4) has a family of solutions, in which one of the parameters (α, ϖ, c) can be chosen arbitrarily (Biktashev, 2003). The family of solutions is

$$U(z) = \begin{cases} -\alpha + \alpha e^{cz} & z \leq z_1 \\ \varpi - \dfrac{a^2c^2}{1+ac^2} e^{(-\frac{z}{ac})} & z \geq z_1 \end{cases} \tag{10}$$

$$V(z) = \begin{cases} 1 & z \leq 0 \\ e^{(-\frac{z}{ac})} & z \geq 0 \end{cases} \tag{11}$$

where

$$\varpi = 1 + ac^2(\alpha + 1), \quad z_1 + \frac{1}{c} \ln\left(\frac{1+\alpha}{\alpha}\right) \tag{12}$$

and speed c is an implicit function of a and α, defined by

$$ac^2 \ln\left(\frac{(1+\alpha)(1+ac^2)}{a}\right) + \ln\left(\frac{1+\alpha}{\alpha}\right) = 0 \tag{13}$$

The structure of this paper is as follows: In Section 2 we introduce in details the simplified telegraph model of the propagation and dissipation of excitation fronts. Section 3 includes exact wavefront solutions in our simplified model and some numerical results. Concluding remarks and discussion are given in Section 4.

2. A SIMPLIFIED TELEGRAPH MODEL OF THE PROPAGATION AND DISSIPATION OF EXCITATION FRONTS

The standard diffusion equation (which is also known as Fick's second law) depends on the continuity equation and the Fick's first law can be written as

$$\frac{\partial u(x,t)}{\partial t} = -\frac{\partial J(x,t)}{\partial t}, \quad J(x,t) = D\frac{\partial u(x,t)}{\partial x} \tag{14}$$

where $J(x, t)$ is the current of the diffusive object, $u(x, t)$ is the distribution function of the diffusing quantity, and D is the diffusion constant. The resulting standard equation is

$$\frac{\partial u(x,t)}{\partial t} = D\frac{\partial^2 u(x,t)}{\partial x^2} \tag{15}$$

A basic weakness of this equation is that the flux $J(x, t)$ reacts instantaneously to the gradient of $u(x, t)$ consequently an unbounded propagation speed is assumed.

A way to see this unphysical effect is to solve the diffusion equation (15) by the use of Fourier and Laplace transforms.

$$u(x,t) = \frac{1}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} c(\tilde{x},0) e^{-\frac{(x-\tilde{x})^2}{4Dt}} d\tilde{x} \tag{16}$$

This equation is also known as the Poisson integral.

The convolution kernel is the Gaussian bell curve with width of $\sqrt{4Dt}$. Consider now an initial condition in the form of a δ–function, $c(x,0) = \delta(x)$.

Then

$$u(x,t) = \frac{1}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}} \tag{17}$$

So a δ–function diffuses out as a Gaussian. If one now considers an arbitrary condition $c(x,0)$ as a sum (integral) of δ–functions, one can see that the solution $c(x,t)$ is the sum (integral) of the diffused δ–functions, which have become Gaussian's. This helps to get an intuitive understanding of Eq. (15). It is obvious that no matter how large x is and how small t may be, then $c(x,t)$ is nonzero which violates the fact that all physical propagation speeds ($\frac{\Delta x}{\Delta t}$) are finite. This is specially true in biological and economical systems where it is known that in many cases propagation speed is typically small. This is, mathematically speaking, due to the fact that Eq. (15) is a parabolic partial differential equation.

To overcome this weakness (Fick's law), Cattaneo in 1948 proposed a modified diffusion equation or (Telegraph equation) (Cattaneo, 1948)

$$\frac{\partial u(x,t)}{\partial t} + \tau \frac{\partial^2 u(x,t)}{\partial t^2} = D \frac{\partial^2 u(x,t)}{\partial x^2} \tag{18}$$

for some constants D and τ.

The corresponding Telegraph reaction diffusion (TRD) equation of (18) is given by (Ahmed and Hassan, 2000)

$$\tau \frac{\partial^2 u(x,t)}{\partial t^2} + \left(1 - \tau \frac{df}{du}\right) \frac{\partial u(x,t)}{\partial t} = D \frac{\partial^2 u(x,t)}{\partial x^2} + f(u) \tag{19}$$

where $f(u)$ represents the reaction term and τ is a time constant, which is a measure of the memory (delay) effect.

Equation (19) can be generalized to the following system (Ahmed, Abdusalam, and Fahmy, 2001):

$$\tau \frac{\partial^2 u_i}{\partial t^2} + \frac{\partial u_i}{\partial t} - \tau \sum_j \frac{\partial u_j}{\partial t} \frac{\partial f_i}{\partial u_j} = D_{il} \frac{\partial^2 u_i(x,t)}{\partial x^2} + f_i(u_1, u_2, \ldots, u_n) \tag{20}$$

where $i, j = 1, 2, \ldots, n$, and D_{ij} are the diffusion constants.

Now, systems (3) and (4) can be generalized to the following simplified telegraph model of propagation and dissipation of excitation fronts as

$$\tau\frac{\partial^2 u}{\partial t^2}+\left(1-\tau\frac{\partial f}{\partial u}\right)\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+f(u,v),\quad f(u,v)=H(u-1)v \quad (21)$$

$$\tau\frac{\partial^2 v}{\partial t^2}+\left(1-\tau\frac{\partial g}{\partial v}\right)\frac{\partial v}{\partial t}=g(u,v),\quad g(u,v)=\frac{1}{a}(H(-u)-v) \quad (22)$$

where $D_{11}=1$ and $D_{22}=0$.

3. TRAVELING WAVEFRONT SOLUTIONS

Also, here we look for traveling wave solution satisfying (8) and (9), systems (21) and (22) can be reduced to

$$(\tau c^2-1)U''+(1-\tau(\delta(U-1)V)cU'=H(U-1)V \quad (23)$$

$$\tau c^2V''+\left(1+\frac{\tau}{a}\right)cV'=\frac{1}{a}(H(-U)-V) \quad (24)$$

By using the internal and auxiliary conditions (8) and (9), the simplified model has a family of solutions given by

$$U(z)=\begin{cases}-\alpha+\alpha\exp\left(\frac{c}{1-\tau c^2}\right)z & z\le z_1\\ \varpi=\frac{a^2c^2}{(\tau c^2-1-ac^2)}e^{(-\frac{z}{ac})} & z\le z_1\end{cases} \quad (25)$$

$$V(z)=\begin{cases}1 & z\le 0\\ e^{-\frac{1}{ac}z} & 0\le z\end{cases} \quad (26)$$

where

$$\varpi=1+\frac{a^2c^2(1+\alpha)}{(1-\tau c^2)},\qquad z_1=\left(\frac{1-\tau c^2}{c}\right)\ln\left(\frac{1+\alpha}{\alpha}\right)=0 \quad (27)$$

and speed c is an implicit function of a, τ, and α, defined by

$$ac^2\ln\left(\frac{(1+\alpha)(\tau c^2-1-ac^2)}{a(\tau c^2-1)}\right)+(1-\tau c^2)\ln\left(\frac{1+\alpha}{\alpha}\right)=0 \quad (28)$$

It can be easily seen that when $\tau=0$ Eqs. (25)–(28) are reduced to (10)–(13).

Note that here the post-front voltage ϖ depends on two parameters, a and the time delay τ; this is different from both the FHN-type system and simplified Biktashev model (Biktashev, 2003). The effect of τ is clear from Fig. 1. The thin graph corresponding to $U(z)$, where $c=0.444$, $\alpha=1$, $a=8$, and $\tau=0.5$. The

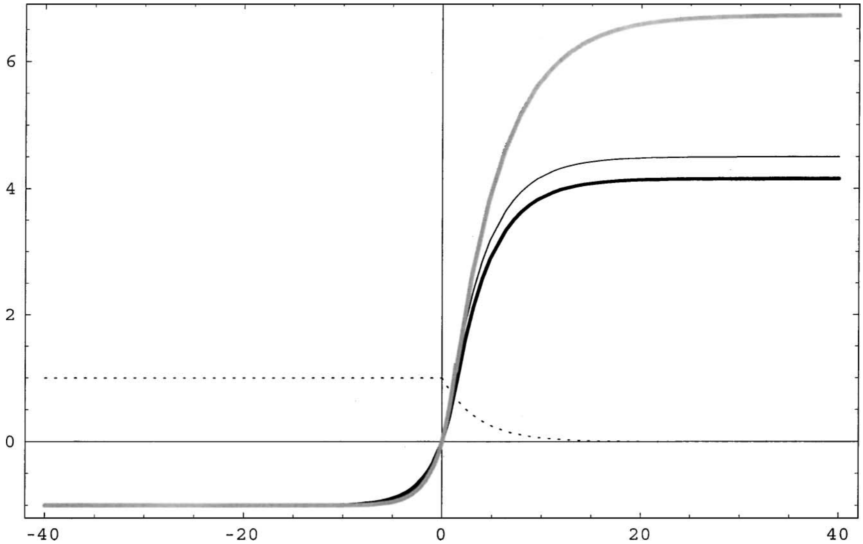

Fig. 1. The thin graph corresponding to $U(z)$ where $c = 0.444$, $\alpha = 1$, $a = 8$, and $\tau = 0.5$. The solid black for the same values of c, α, and a but with $\tau = 0$. The solid gray graph for $c = 0.5$, $\alpha = 1$, $a = 10$, and $\tau = 0.5$. The graph of $V(z)$ is represented by dots.

same values of c, α, a are used for the solid black graph but with $\tau = 0$. The values $c = 0.5$, $\alpha = 1$, $a = 10$, and $\tau = 0.5$ are used for solid gray graph. The graph of $V(z)$ is represented by dots.

4. CONCLUSIONS

In our present work, we present a new simplified telegraph model of propagation and dissipation of excitation fronts. An exact analytic traveling wave front solutions for our generalized model are obtained. The effect of time delay which appears in the telegraph system is presented by a figure for different values, where for some values of the time delay the work is still open for further research work.

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